

Deriving the Regge-Wheeler and Zerilli equations in the general static spherically-symmetric case with MathematicaTM and MathTensorTM

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February 6, 2008

Abstract

An efficient approach to tensor perturbation calculations by proper use of computer algebra methods is described, reaching the sufficient generality required for a comprehensive analysis of the Schwarzschild and Reissner-Nordström metric stability problem.

1 Introduction

The stability problem for a Schwarzschild black hole in the form of a “pure metric” perturbation analysis was settled by T. Regge and J. A. Wheeler in a classic 1957 article [1]. The main practical achievement of this work was undoubtedly the formulation of a gauge transformation approach that allows a complete radial/angular separation of the Einstein equations in the two cases of odd and even parity, preliminarily established, reaching a formal solution in the axial (or *magnetic*) case: the so-called *Regge-Wheeler Equation*. Due to mathematical complications, however, the full analysis was only completed thirteen years later, after the work of Mathews [2], Edelstein-Vishveshwara [3] and Zerilli [4] that either provided a more rigorous approach to the use of tensor harmonics, or resolved some compatibility problems in the analytic treatment of the system, or else provided the final form of the equation for the radial perturbation functions in the polar (or *electric*) case: the so-called *Zerilli Equation*. The search for simplicity also led these authors to exploit some useful but not general relations between curvature tensors, like those derived by Eisenhart [5], valid to the first order and/or only in the Schwarzschild case (latin tensor indices are used for consistency with the implemented algorithms):

$$\delta G_{mn} = \delta R_{mn} \quad (1)$$

$$\delta R_{mn} = \delta \Gamma_{mn;p}^p - \delta \Gamma_{mp;n}^p \quad (2)$$

$$\delta \Gamma_{jk}^i = \frac{1}{2} g^{ip} (h_{jp;k} + h_{kp;j} - h_{jk;p}) \quad (3)$$

where h is the perturbation tensor and $\delta \Gamma$, δR , δG are the perturbed parts of the affine connections, Ricci and Einstein tensors.

On this way, after having performed the gauge transformations for each parity case, we are left with two systems of, respectively, three and six independent radial ordinary differential

equations in two (namely $h_0(r)$, $h_1(r)$) and three (namely $H(r)$, $H_1(r)$, $K(r)$) arbitrary perturbation functions, to be determined. The magnetic system is straightforwardly reduced to a single first-order equation in $h_1(r)$ which, by a simple variable substitution, leads to the final result:

$$Q''(r^*) + [k^2 - V(r)]Q(r) = 0 \quad (4)$$

(*Regge-Wheeler Equation*), where r^* and Q (the first called the *tortoise coordinate* by Wheeler as a citation of the Zeno paradox) are (implicitly or not) defined by:

$$d/dr^* = a(r)d/dr \quad Q(r) = b(r)h_1(r) \quad (5)$$

where $a(r)$, $b(r)$ are arbitrary functions to be determined in each particular case and $V(r)$, in this Schrödinger-like equation, plays the role of an effective potential.

Finally, the electric system, by a more complex change of variables procedure, necessary to deal with the terms in the wave number k , was found by Zerilli [4] to be represented by an equation formally equal to (4) (the *Zerilli Equation*) only with a different (but still algebraic) expression of the potential. Since all the variables of the system are mutually expressed by regular algebraic relations, the stability problem, analyzed by substituting different forms of k into (4), can be extended in its validity to the whole perturbation. The aim of the present work is to show that, with the essential help of computer algebra software, a similar analysis can be carried out for both parity cases, dealing with the more general spacetime of a spherical, non-rotating, eventually charged collapsed object, therefore allowing specialization not only to the Schwarzschild but also to the Reissner-Nordström metric.

2 The static spherically-symmetric system

To begin a less rigid analysis than that induced by the formulation of equations (1–3), we deal first of all with the full expression of the Einstein tensor which, viewed as a function of the metric and its ordinary partial derivatives, reads as:

$$\begin{aligned} \text{In[2]:= } G_{mn} &= \text{Simplify}[\text{Timplify}[\text{AffineToMetric}[\text{RicciToAffine}[\text{RicciR}[\text{lm}, \text{ln}]]] - \\ &\quad \text{1/2 Dum}[\text{AffineToMetric}[\text{ScalarRtoAffine}[\text{ScalarR}]] \text{Metricg}[\text{lm}, \text{ln}]]] \\ \text{Out[2]= } &\frac{1}{8} (4 ((g_{pm,n} + g_{pn,m} - g_{mn,p}) (g^{pq,q}) - (g_{pq,m}) (g^{pq,n})) + \\ &\quad (g_{mn}) (g^{pq}) (2 ((g_{rs,q}) (g^{rs,p}) + (g_{pq,r} - g_{pr,q} - g_{qr,p}) (g^{rs,s})) + \\ &\quad (g^{rs}) ((g^{tu}) ((g_{qr,t} + g_{qt,r} - g_{rt,q}) (g_{ps,u} - g_{pu,s} + g_{su,p}) + \\ &\quad (g_{qr,s} + g_{qs,r} - g_{rs,q}) (g_{pt,u} - g_{pu,t} - g_{tu,p})) + \\ &\quad 2 (g_{pq,rs} - g_{ps,qr} - g_{qr,ps} + g_{rs,pq})) - \\ &\quad 2 (g^{pq}) ((g^{rs}) ((g_{ps,n}) (g_{qr,m}) + 2 (g_{pn,s}) (- (g_{qm,r} + g_{rm,q}) - (g_{rs,p}) (g_{qm,n} + g_{qn,m} - g_{mn,q})) + \\ &\quad 2 (g_{pq,mn} - g_{pm,qn} - g_{qn,pm} + g_{mn,pq}))) \end{aligned}$$

Here, to the unperturbed metric, which refers to the usual covariant expression of the line element belonging to a generic spherically-symmetric expression depending on the radial arbitrary functions $\lambda(r)$, $\nu(r)$:

$$ds^2 = e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - e^{\nu(r)} dt^2$$

must be added the perturbation terms, whose time dependency is represented only by a e^{-ikt} factor while the angular one is limited to functions of the polar angle θ .

Since ”**Components**”, MathTensor’s routine for calculating curvature tensors from an input metric, has the useful feature of applying arbitrary combinations of Mathematica commands to each component [7], the first order form of the Einstein tensor can be readily obtained by appending to the metric input file, where the Regge-Wheeler-gauge perturbation terms are added with a “small parameter” q as a factor, a line like:

```
CompSimp[a_]:=Simplify[Normal[Series[Expand[a],{q,0,1}]]/.q->1]
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2.1 The magnetic case

There are only three non-zero components $\{(r, \phi), (\theta, \phi), (\phi, t)\}$ of the tensor equation that replaces the (1):

$$\delta G_{mn} = G_{mn}^{q=1} - G_{mn}^{q=0} = 0 \quad (6)$$

(where the subtracted quantity is the unperturbed metric tensor) and that substantially reproduce, in their radial form, the solution system [3]-(sys. 2), once the angular terms coming from the perturbation, where they are represented by $f(\theta) = \sin \theta \partial P_L(\cos \theta) / \partial \theta$ (P_L being the Legendre polynomial to the multipolar order L), are fully simplified through the following relations:

$$\begin{aligned} f(\theta) &= L [-P_{L-1}(\cos \theta) + L \cos \theta P_L(\cos \theta)] \\ f'(\theta) &= -L(L+1) \sin \theta P_L(\cos \theta) \\ f''(\theta) &= L(L+1) [L P_{L-1}(\cos \theta) - (L+1) \cos \theta P_L(\cos \theta)] \end{aligned}$$

and, being completely factored, are consequently eliminated.

The first two equations (from now on, almost everywhere in the rest of this paper, the notation $\lambda = (L-1)(L+2)/2$ will be adopted -not to be confused with the definition of the metric’s radial function $\lambda(r)$!) turn out to be of the first order with non-mixed dependence on the functions’ derivatives in such a way that a single second-order differential equation can be obtained by:

```
In[12]:= Eq1 = 4 I E^{-\lambda[x]} k r h_0[r] +
  (4 - 4 (\lambda + 1) + 2 E^{-\lambda[x]} k^2 r^2 + E^{-\lambda[x]} r ((\lambda'[x] - v'[x]) (2 + r v'[x]) - 2 r v''[x])) h_1[r] -
  2 I E^{-\lambda[x]} k r^2 h_0'[r] == 0;
Eq2 = -2 I E^{\lambda[x]} k h_0[r] + E^{\lambda[x]} (\lambda'[x] - v'[x]) h_1[r] - 2 E^{\lambda[x]} h_1'[r] == 0;
s = Flatten[Solve[Eq2, h_0[r]]];
ds = D[s, x];
p = Expand[First[FullSimplify[Eq1 /. Flatten[{s, ds}]]]];
cf = Coefficient[p, h_1''[r]]; cf != 0
EqM = Collect[Expand[Simplify[p / cf]], {h_1''[r], h_1'[r], h_1[r]}, FullSimplify] == 0
```

```
Out[12]= 2 E^{-\lambda[x]} r^2 != 0
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Out[13]= (-\frac{2}{r} - \frac{3 \lambda'[x]}{2} + \frac{3 v'[x]}{2}) h_1'[r] + \frac{1}{2 r^2} (h_1[r]
  (2 E^{\lambda[x]} (E^{-\lambda[x]} k^2 r^2 - 2 \lambda) + r (r \lambda'[x]^2 - 4 v'[x] + \lambda'[x] (4 - r v'[x]) - r (\lambda''[x] + v''[x]))) +
  h_1''[r]) ==
0
```

Analyzing the dependence on k , it is readily seen that to impose the equation (4) with the conditions (5), a procedure of polynomial coefficient identification is required, carried out in the following way:

```

In[14]:= Eq = Expand[(D_x (D_x (b[r] h1[r])) a[r]) a[r]] + (k^2 - V[r]) b[r] h1[r];
SimbEq = Collect[Eq / Coefficient[Eq, h1''[r]], {h1[r], h1'[r], h1''[r]}, Together] == 0

Out[14]= (b[r] a'[r] + 2 a[r] b'[r]) h1'[r] +
          a[r] b[r]
          (h1[r] (k^2 b[r] - b[r] V[r] + a[r] a'[r] b'[r] + a[r]^2 b''[r])
           a[r]^2 b[r]) + h1''[r] ==
0

In[15]:= Tort = Solve[Coefficient[Coefficient[First[SimbEq], h1[r]], k^2] ==
      Coefficient[Coefficient[First[EqM], h1[r]], k^2],
      a[r]]

Out[15]= {{a[r] -> -E^(1/2 (-lambda[r] + gamma[r]))}, {a[r] -> E^(1/2 (-lambda[r] + gamma[r]))}}

In[16]:= a[r_] = Last[Tort][[2, 1]]; CoeffH1 =
      DSolve[Coefficient[First[SimbEq], h1'[r]] == Coefficient[First[EqM], h1'[r]], b[r], r]

Out[16]= {{b[r] -> -sqrt(E^((1/2) (-lambda[r] + gamma[r])))/r}, {b[r] -> sqrt(E^((1/2) (-lambda[r] + gamma[r])))/r}}

In[17]:= b[r_] = Last[CoeffH1][[2, 1]] /. C[1] -> 0; FullSimplify[
      Solve[Coefficient[First[EqM], h1[r]] == Coefficient[First[SimbEq], h1[r]], V[r]]]

Out[17]= {{V[r] -> (E^(-lambda[r] + gamma[r]) (4 + 4 E^lambda[r] lambda[r] + r (-lambda'[r] - gamma'[r]) (1 + r gamma'[r]) + 2 r gamma''[r])) / (2 r^2)}}

```

2.2 The electric case

Here we deal with the seven non-zero components of the perturbed Einstein tensor (the complementary set with respect to the previous three “magnetic” components), whose angular parts, coming from the perturbing gauge function $F(\theta) = P_L(\cos \theta)$, are factored, once the substitutions

$$F'(\theta) = \frac{L}{\sin \theta} [-P_{L-1}(\cos \theta) + \cos \theta P_L(\cos \theta)]$$

$$F''(\theta) = \frac{L \cos \theta}{\sin^2 \theta} P_{L-1}(\cos \theta) - L (\cot^2 \theta + L + 1) P_L(\cos \theta)$$

are performed, the resulting expressions simplified and the original four radial perturbation functions are reduced to three by the identification $H_0(r) = H_2(r) \equiv H(r)$.

Of the six independent linear differential equations so obtained (the diagonal θ and ϕ terms being equal), four turn out to be of the first order, unlike the Schwarzschild-specialized system which instead has three first order equations, and a completely algebraic variable-elimination procedure allows the derivation of a first integral condition which is the generalization of

[3]-(eq. 10). As in Zerilli's procedure [6], more elaborate but always tied to a polynomial coefficient identification principle, this algebraic condition plus three of the previous equations can be treated, with a double function substitution and the change of the radial coordinate, to form a new system of seven differential equations in four variables (plus the derivative of the new radial coordinate with respect to r), this time non-linear but independent of k , to which the formal definition of the effective potential must be added. Three of these unknown functions are then found, after a cascade of algebraic eliminations (which don't show the residual arbitrariness found by Zerilli in the Schwarzschild treatment), to be quite simply dependent on the fourth, which satisfies a final second-order very complex equation, fortunately analytically solvable when specialized to the Schwarzschild and Reissner-Nordström metrics.

3 Results of the analysis

A comprehensive table of functions and variables in the three cases – general static spherically-symmetric form (G), Schwarzschild (S) and Reissner-Nordström (RN) – can be sketched, the latter two obtained, respectively, by the two substitutions $\{\lambda(r) = -\ln(1 - 2m/r) ; \nu(r) = -\lambda(r)\}$ and $\{\lambda(r) = -\ln(1 - 2m/r + Q^2/r^2) ; \nu(r) = -\lambda(r)\}$:

π	<i>Schrödinger wave-like function</i>	<i>Tortoise coordinate</i>
G	mag. $Q(r) = \exp\left[\frac{1}{2}(\nu(r) - \lambda(r))\right] \frac{h_1(r)}{r}$	$r^* = \int \exp\left[\frac{1}{2}(\lambda(r) - \nu(r))\right] dr$
	el. $\hat{K}(K(r), \frac{H_1(r)}{r})$	$r^* = \int e^{\frac{\lambda(r)}{2}} \left[\frac{2\lambda e^{\lambda(r)} - r\lambda'(r) + 2r\nu'(r)}{e^{\nu(r)}(2\lambda e^{\lambda(r)} + 3r\nu'(r))} \right]^{\frac{1}{2}} dr$
S	mag. $Q(r) = \left(1 - \frac{2m}{r}\right) \frac{h_1(r)}{r}$	$r^* = r + 2m \ln(r - 2m)$
	el. $\hat{K}(K(r), \frac{H_1(r)}{r})$	
RN	mag. $Q(r) = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) \frac{h_1(r)}{r}$	$r^* = r + m \ln(r^2 - 2mr + Q^2) + \eta(r)$
	el. $\hat{K}(K(r), \frac{H_1(r)}{r})$	$\left[\eta(r) = \frac{2m^2 - Q^2}{\sqrt{Q^2 - m^2}} \arctan\left(\frac{r - m}{\sqrt{Q^2 - m^2}}\right) \right]$

and the correspondent couples of specialized expressions of the Regge-Wheeler and Zerilli's potentials are:

- Schwarzschild:

$$V_{mag}^S(r) = 2 \left(1 - \frac{2m}{r}\right) \left(\frac{\lambda + 1}{r^2} - \frac{3m}{r^3}\right)$$

$$V_{el}^S(r) = 2 \left(1 - \frac{2m}{r}\right) \frac{\lambda^2(\lambda + 1)r^3 + 3\lambda^2mr^2 + 9\lambda m^2r + 9m^3}{(\lambda r + 3m)^2 r^3}$$

- Reissner-Nordström:

$$V_{mag}^{RN}(r) = 2 \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) \left(\frac{\lambda + 1}{r^2} - \frac{3m}{r^3} + \frac{3Q^2}{r^4} \right)$$

$$V_{el}^{RN}(r) = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) \frac{P(r)}{4 [r(\lambda r + 3m) - 3Q^2]^2 r^4}$$

with

$$P(r) = 8\lambda^2(\lambda + 1)r^6 + 24\lambda^2mr^5 + 2\lambda [36m^2 - 7(2\lambda - 3)Q^2] r^4$$

$$+ 12m [6m^2 - (19\lambda - 3)Q^2] r^3 - 3Q^2 [108m^2 - (38\lambda - 3)Q^2] r^2$$

$$+ 342mQ^4r - 117Q^6$$

As a final remark, it is straightforward to verify that the two different kinds of tortoise coordinates in the general case reduce to one for all metrics having $\lambda(r) = -\nu(r)$ and that as expected, $\lim_{Q \rightarrow 0} V^{RN}(r) = V^S(r)$ holds for both the magnetic and the electric parities.

References

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